Real Analysis qualification exam: August 2018

Note: all statements require proofs. You can make references to standard theorems from the course, but you need to specify which exact standard fact you are referring to. $m(E)$ always denotes the Lebesgue measure of E.

(1) Let $A \subset (1, +\infty)$ satisfy the following property: $x \in A$ if and only if $x^2 \in A$ (assuming that $x > 1$). Suppose also that A is Lebesgue measurable. Find all possible values of $m(A)$.

(2) Let $f, g, h \in L^3(X, \mu)$, where $(X, \mathcal{B}, \mu)$ is a measure space. Show that

$$\left(\int_X |fgh|\, d\mu\right)^3 \leqslant \left(\int_X |f|^3\, d\mu\right)\left(\int_X |g|^3\, d\mu\right)\left(\int_X |h|^3\, d\mu\right)$$

(3) Let $f \in L^1(\mathbb{R})$. Suppose that $\{E_i\}_{i=1}^{+\infty}$ is a sequence of measurable sets, $E_i \supset E_{i+1}$, $m(E_1) < +\infty$. Let $E = \cap_i E_i$. Show that

$$\lim_{n\to\infty} \int_{E_n} f\, dm = \int_E f\, dm?$$

Would the answer change if $f\colon \mathbb{R} \to [0, +\infty)$ is a measurable function, not necessarily from $L^1(\mathbb{R})$? Justify your answer.

(4) Let $f\colon [0,1] \to (0, +\infty)$ be an absolutely continuous function, such that $f(x) > 0$ for all $x \in [0,1]$. Is it true that $1/f$ is absolutely continuous on $[0,1]$? Justify your answer.

(5) Does there exist a closed subset $A \subset (0,1)^2$ of positive measure such that, for all $(x, y) \in A$, both x and y are irrational? Justify your answer.